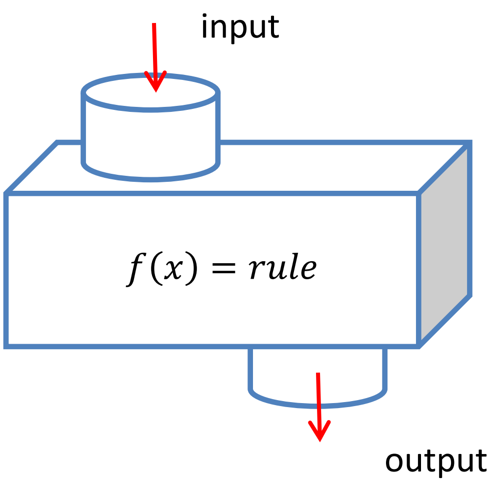
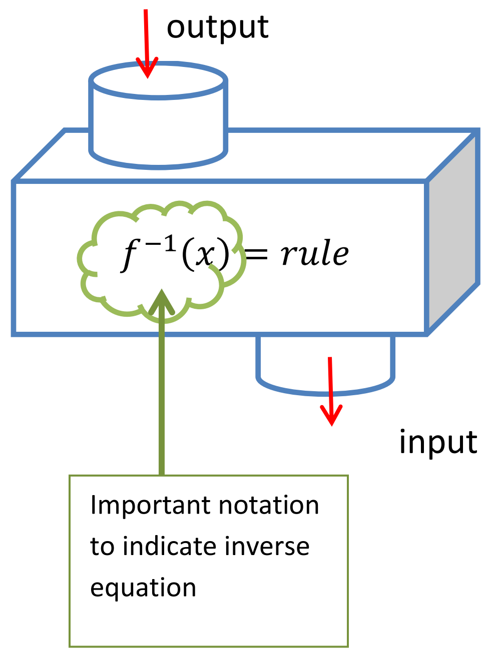
INVERSE OF FUNCTIONS

Most of the time in math the input is given and one is asked to find the corresponding output value. However, in real life applications the target output value is known but the corresponding input value is not. For example, if one starts saving money in a bank account that pays interest every month, and the person wants to know how long it will take to save a given amount of money. This situation can be modeled by using the inverse equation of an exponential. The main idea behind an inverse is to "undo" either basic operations (addition, subtractions,…) or more complex operations. It often helps to think of inverse also as working backwards.

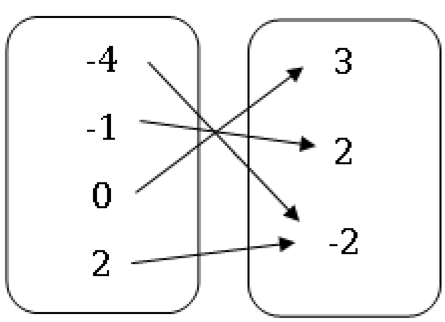
Visual Representations of Inverse Functions

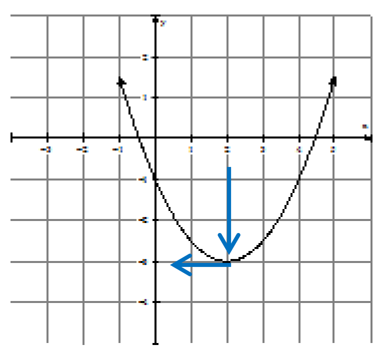




Example:

x	$f(x)$
-2	4
1	-1
3	-6
4	3





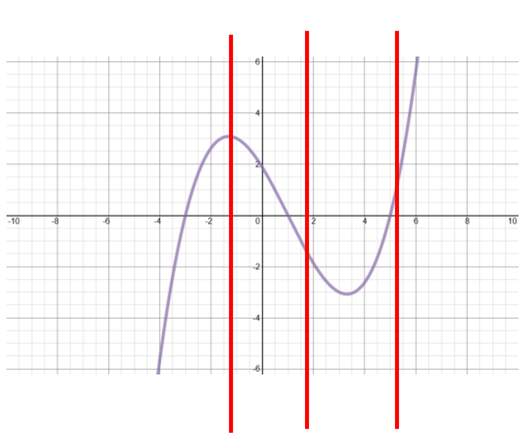
$f(1) = -1 \text{ so } f^{-1}(-1) = 1$ $f(0) = 3 \text{ so } f^{-1}(3) = 0$ $f(2) = -3 \text{ so } f^{-1}(-3) = 0$

$$x \rightarrow y \text{ so inverse means } y \rightarrow x$$

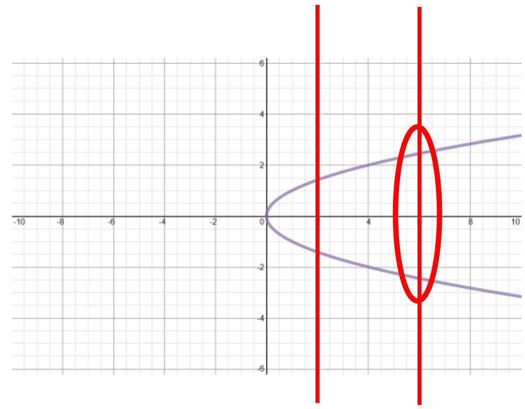
!!Be careful not to violate the definition of being a function; inverses have the ability to do that!!!

Functionality

Recall that a function is a relation where each element of the domain corresponds to a unique (only one) element of the range. When given a function represented graphically, the vertical line test would be used to make sure that each value of x only corresponded to one value of y.

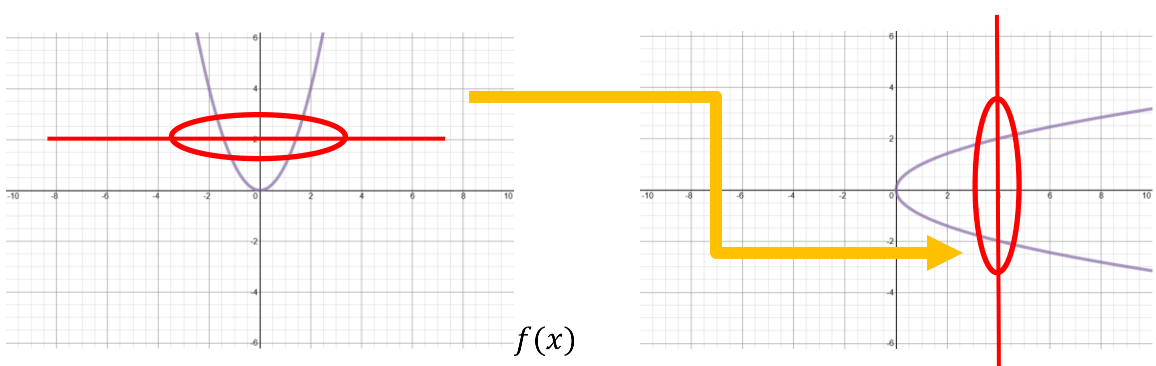


The graph above is a function since each vertical line only touches the graph once for each value of x.

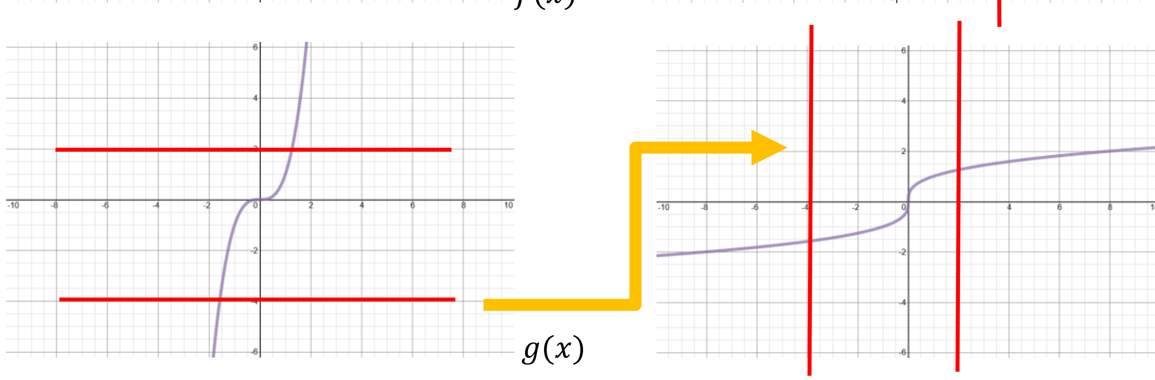


At $x = 0$ the graph will pass the vertical line test, but notice for example at $x = 4$ the vertical line touches the graph twice at the values $y = 2 \text{ and } y = -2$. Since the vertical line touches twice the graph above does not represent a function.

Now if we combine the idea above of switching $x \text{ with } y$ and the vertical line test, we get the horizontal line test. Since the equation of a vertical line is $x = \#$ then we switch $x \text{ with } y$ we get the equation of a horizontal line $y = \#$. The horizontal line test is done on the original function $f(x)$, if it only touches the graph once at every y-value then the inverse of $f(x)$ will be a function.



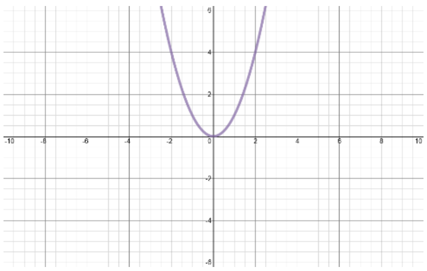
EX1 $f(x)$ $f^{-1}(x)$



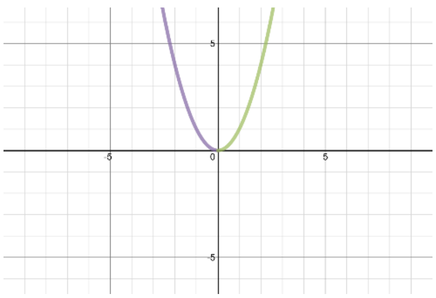
EX2 $g(x)$ $g^{-1}(x)$

Notice that in example 1, $f(x)$ failed the horizontal line test which mean that $f^{-1}(x)$ failed the vertical line test so $f^{-1}(x)$ is NOT a function. However, in example 2 $g(x)$ passed the horizontal line test which means $g^{-1}(x)$ is a function. We call these function **one-to-one.**

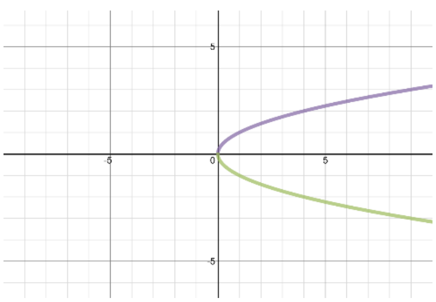
Say we have a relations, $h(x)$, which passed the vertical line test so it is a function but did not pass the horizontal line test. [Note: this means $h(x)$ is NOT one-to-one]. We can make $h(x)$ a one-to-one function by restricting the domain. Let us take a look at this visually.



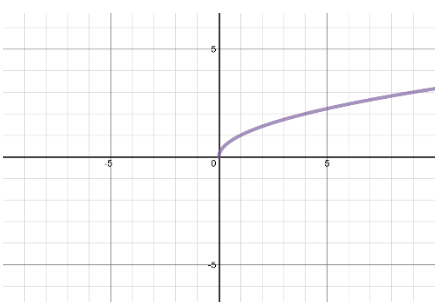
This is a quadratic and it is a function. However, it does not pass the horizontal line test so the inverse is not a function.



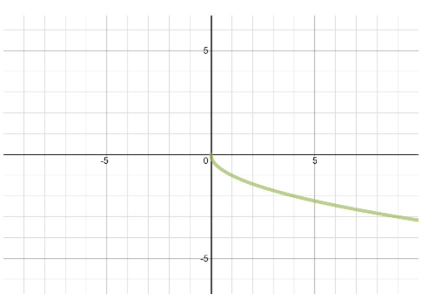
This problem can be fixed by restricting the domain of the quadratic. If we restrict the domain $\{x \mid x \leq 0\}$ *or* $\{x|x \geq 0\}$, this allows us to break the function into two part. Each part will now pass the horizontal line test make the inverse a function, and it also makes the function one-to-one.



So here is the graph of the inverse that shows the separation of the two parts. Now we can remove one of the parts depending on the problem.



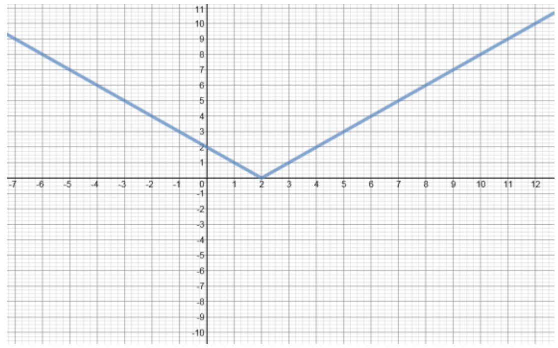
Here only the top of the inverse is being considered, when $y \geq 0$.



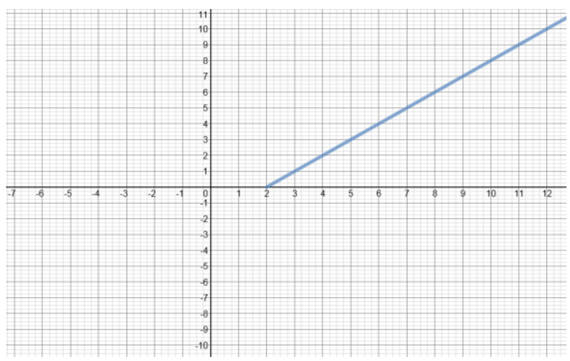
Here only the bottom is being considered, when $y \leq 0$.

Example of Domain Restriction

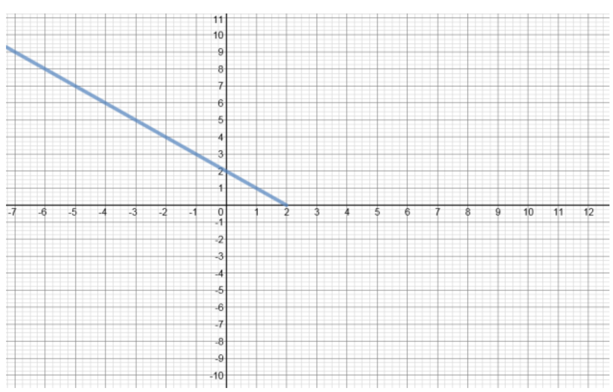
The function, $f(x) = |x - 2|$ is not a one-to-one function because it does not pass the horizontal line test. However, we can restrict the domain of $f(x)$ so that it now will pass the horizontal line test.



Suppose we define $f(x) = |x - 2|$ where $x \geq 2$. Now we only are considering half of the graph but this allows us to now create an inverse. We would write a linear equation for this line and then algebraically find the inverse.



We could also restrict the domain where $f(x) = |x - 2|$ where $x \leq 2$. We then again would write an equation for this linear and then algebraically find the inverse of the function now.



Finding the Inverse Algebraically

The mathematical process of writing the inverse of a function is better illustrated through an example. Warning: this works on basic, less complex functions. The more complex a function and its operations the more creative of algebraic processes that would be required.

Consider the function, $g(x) = -2(x+1)^3 - 7$, this equation will offer a little bit of everything.

$$g(x) = -2(x+1)^3 - 7$$

- Start with the given

$$y = -2(x+1)^3 - 7$$

- To help make this process easier, it is suggested to change the function notation into terms of x and y.

$$x = -2(y+1)^3 - 7$$

- Now using the main idea of inverses switch x and y. Start isolating or solving for y. Here is where many mistakes can be made; the key is to remember your order of operations and work backwards through them.

$$x + 7 = -2(y+1)^3$$

- So if the order of operation is remembered by PEMDAS. To begin we must start with "AS", that is any addition and subtraction that is not held in parentheses

$$\frac{x+7}{-2} = (y+1)^3$$

- So if the order of operation is remembered by PEMDAS. To begin we must start with "MD", that is any multiplication and division that is not held in parentheses

$$\sqrt[3]{\frac{x+7}{-2}} = y + 1$$

- "E" exponents → undoing exponents required the use of radicals (square roots, cube roots, ...).

$$\sqrt[3]{\frac{x+7}{-2}} - 1 = y$$

- "P" parentheses → to work here you start all over with the order of operation backwards for what is held in the parentheses.

$$g^{-1}(x) = \sqrt[3]{\frac{x+7}{-2}} - 1$$

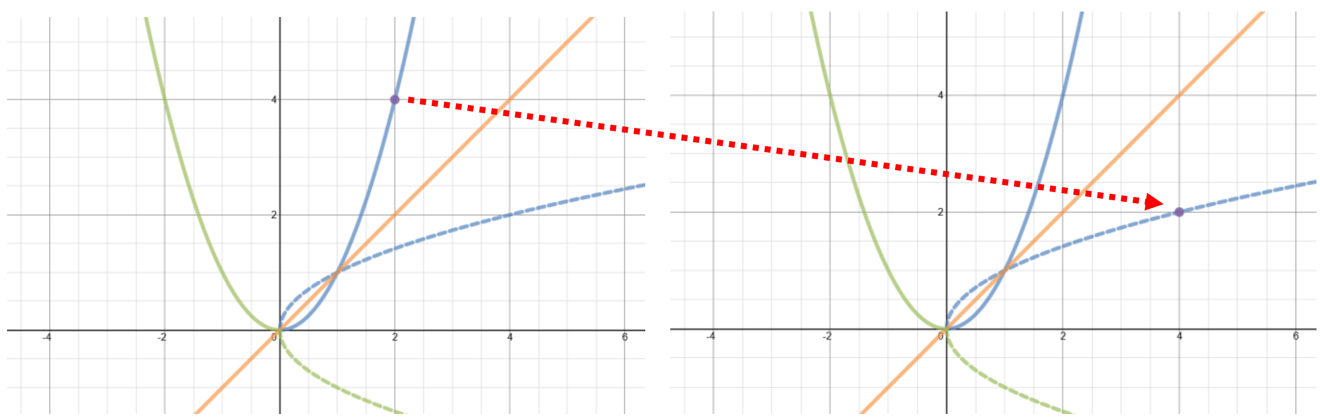
- Once y is isolated we rename it as an inverse to distinguish it from the original.

Graphing Inverse Functions

There are two ways to create the graph of an inverse function.

Given $f(x)$	
Create a table of values for $f(x)$	Find the inverse equation of $f(x)$
Switch all x value with the corresponding y value so that x-values are now the output and y-values are now the input	Graph inverse equation Or Make a table of value for inverse equation
Plot points and graph	Plot points and graph

Big Key Connection: The graph of the original function and the graph of the inverse are reflection over the line $y = x$.



The original function is represented by the solid graph where the inverse is represented by the dotted. Notice that the point (2,4) on $f(x)$ *corresponds to* (4,2) *on* $f^{-1}(x)$.

IMPORTANT NOTE: Since the x *and* y values are switch this means that the domain of the original function becomes the range of the inverse. The range of the original becomes the domain of the invers.

$$f(x) = \sqrt{x+1} \quad d: \{x|x \geq -1\} \quad r: \{y|y \geq 0\}$$

$$f^{-1}(x) = x^2 - 1 \quad d: \{x|x \geq 0\} \quad r: \{y|y \geq -1\}$$