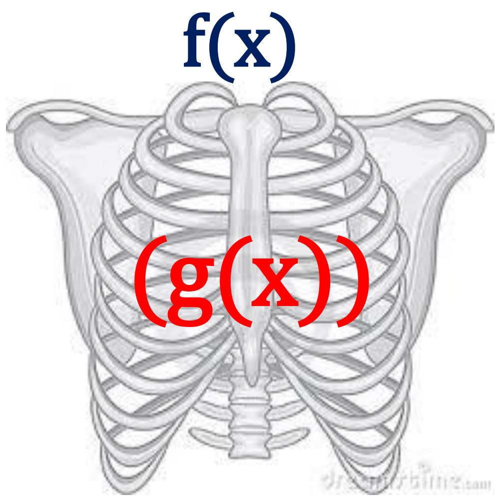
COMPOSITE FUNCTIONS

A composite function may be thought of as a function within a function and may be represented in the following forms: $f \circ g(x)$ or $f(g(x))$, and is pronounced " f of g of x". In essence you can think of it with a "Skeleton and Guts" representation. In the composite function $f(g(x))$ the outer function $f(x)$ represents the skeleton and the inner function $g(x)$ represents the guts.

Visual Representations of Composite Functions

$f \circ g(x)$ or $f(g(x))$



Consider the equations $f(x) = x^2 + 3x - 7$ and $g(x) = 5x + 9$. To represent the function $f(g(x))$, we first set up the skeleton by replacing all of the x's in $f(x)$ with a set of parentheses. This gives us

$$(\qquad\qquad)^2 + 3(\qquad\qquad) - 7$$

We then fill the parentheses (or skeleton if you will) with $g(x)$ (the guts).
Thus:

$$f(g(x)) = (5x+9)^2 + 3(5x+9) - 7$$

to finish the problem just follow order of operations and combine like-terms.

$$(5x+9)^2 + 3(5x+9) - 7$$
$$= (5x+9)(5x+9) + 3(5x+9) - 7$$
$$= (25x^2 + 45x + 45x + 81) + 15x + 27 - 7$$
$$= 25x^2 + 90x + 81 + 15x + 27 - 7$$
$$= 25x^2 + 105x + 101$$

Therefore: $f(g(x)) = 25x^2 + 105x + 101$

Commutative Property

Addition and multiplication are commutative operations $(7+2=2+7 \textit{ and } 3\cdot 4=4\cdot 3)$, basically meaning that order doesn't matter when you are adding two quantities or multiplying two quantities. Subtraction and division are not commutative, however, as order does matter. $\left(5-2\neq 2-5 \textit{ and } \frac{8}{2}\neq\frac{2}{8}\right)$

Let's explore whether or not composition of functions is commutative.
Let $f(x)=7x-9 \textit{ and } g(x)=4x+3$. If the operation of composition is commutative, then $f\circ g(x)$ should equal $g\circ f(x)$.

For $f\circ g(x)$

$f\circ g(x)=f(g(x))$

$=7(\qquad\qquad)-9$ ← Set up the skeleton for the outer function, $f(x)$

$=7(4x+3)-9$ ← Fill in the guts, $g(x)$

$=28x+21-9$ ← Follow order of operations to evaluate

$=28x+12$

For $g\circ f(x)$

$g\circ f(x)=g(f(x))$

$=4(\qquad\qquad)+3$ ← Set up the skeleton for the outer function, $g(x)$

$=4(7x-9)+3$ ← Fill in the guts, $f(x)$

$=28x+36+3$ ← Follow order of operations to evaluate

$=28x+39$

Since $f\circ g(x)\neq g\circ f(x)$, we know that composition is not commutative.

- Note: The only time that composition is commutative is when you are dealing with inverse functions.

- Theorem: Two functions $f(x)$ *and* $g(x)$ are inverses if and only if $f \circ g(x) = g \circ f(x) = x$

- Let's examine this theorem in the following example. Let $f(x) = 8x - 13$.

A. Find $f^{-1}(x)$

Solution:

$f(x) = 8x - 13$

$y = 8x - 13$

$x = 8y - 13$

$x + 13 = 8y$

$\dfrac{x+13}{8} = y$

So $f^{-1}(x) = \dfrac{x+13}{8}$

B. Let $g(x)$ represent the function we just found for $f^{-1}(x)$. So, $g(x) = \dfrac{x+13}{8}$.

Find $f \circ g(x)$ and $g \circ f(x)$

Solution:

$f \circ g(x)$

$= 8(\quad\quad) - 13$

$= 8\left(\dfrac{x+13}{8}\right) - 13$

$= (x + 13) - 13$

$= x$

$g \circ f(x)$

$= \dfrac{(\quad\quad\quad) + 13}{8}$

$= \dfrac{(8x - 13) + 13}{8}$

$= \dfrac{8x}{8}$

$= x$

So, since $f \circ g(x) = g \circ f(x) = x$, then $f(x)$ *and* $g(x)$ are inverses of each other and the commutative property holds.

- Note: This does not constitute a proof that the commutative property for composition of functions only works when the functions are inverses, but it does provide an example in support of the fact. You may feel free to look up the proof online, if you are curious.

Example: For $f(x) = 3x^2 - 7x + 8$ and $g(x) = 4x - 5$, find:

A. $f \circ g(2)$

Solution:

$f \circ g(2) = f(g(2))$

$g(2) = 4(2) - 5 = 8 - 5 = 3$

$= f(3)$

$= 3(3)^2 - 7(3) + 8$

$= 3(9) - 21 + 8 = 27 - 21 + 8 = 6 + 8 = 14$

So, $f \circ g(2) = 14$

B. $g \circ f(-1)$

Solution:

$g \circ f(-1) = g(f(-1))$

$f(-1) = 3(-1)^2 - 7(-1) + 8 = 3(1) + 7 + 8 = 18$

$= g(18)$

$= 4(18) - 5 = 72 - 5 = 67$

So, $g \circ f(-1) = 67$

C. $g \circ f(x)$

Solution:

$g \circ f(x) = g(3x^2 - 7x + 8)$

$= 4(3x^2 - 7x + 8) - 5$

$= 12x^2 - 28x + 32 - 5 = 12x^2 - 28x + 27$

So, $f \circ g(2) = 12x^2 - 28x + 27$

D. $f \circ g(x)$

Solution:

$f \circ g(x) = f(4x - 5)$

$= 3(4x - 5)^2 - 7(4x - 5) + 8$

$= 3(4x - 5)(4x - 5) - 7(4x - 5) + 8$

$= 3(16x^2 - 20x - 20x + 25) - 28x + 35 + 8$

$= 48x^2 - 60x - 60x + 75 - 28x + 35 + 8$

$= 48x^2 - 148x + 118$

So, $f \circ g(x) = 48x^2 - 148x + 118$

Composition by Tables

Given the tables for $f(x)$ *and* $g(x)$ below, find the values of $g(f(1))$ and $g(f(5))$

X	f(x)
1	3
2	1
3	0
4	-1
5	2

x	g(x)
-1	8
0	2
1	5
2	6
3	-4

<u>Solution</u>:

To find evaluate $g(f(1))$, we need to work from the inside out. We will therefore evaluate $f(1)$ first. To do so, we will look at the table for $f(x)$ and find the value of the function when x = 1. So $f(1)=3$. This value then gets plugged into the function $g(x)$. So $g(f(1))= g(3)$. Next, we will evaluate $g(3)$ by examining the value you get from the table for $g(x)$ when x = 3, where we get the value of -4. Therefore, $g(f(1))=-4$.

Another way to look at that question is by noting that the value of x = 1 gets plugged into the innermost function, $f(x)$. The result you get from doing so gets plugged into the outer function, $g(x)$.

Now try $g(f(5))$. First find the value of $f(5)$ from the table. You should get $f(5)=2$. Now use that value to evaluate the function $g(x)$, from the table where x = 2. You should get that $g(2)=6$. So $g(f(5))=6$.

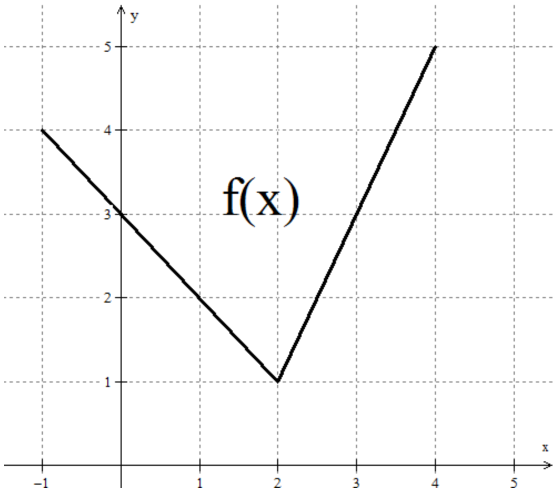
Now Let's try something a bit more challenging.

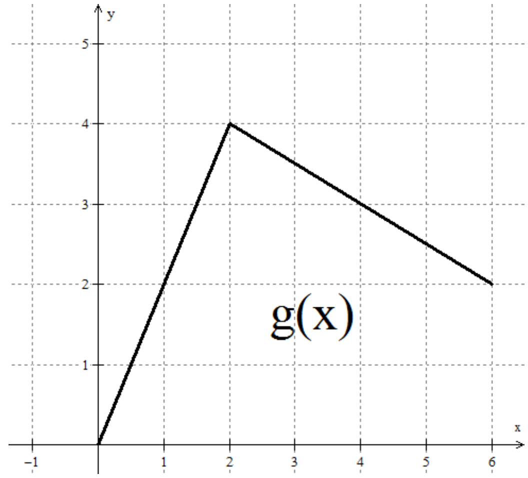
<u>Example</u>: Find the value of x that satisfies the equation $g(f(x))=2$.

In this case, we are not given a value of x to use in the table. We must therefore go backwards from what we did in the previous problems and work from the outside in. First we need to find a value of x in the table $g(x)$ such that $g(x)=2$. Upon doing so, you should get a value of x = 0. We then look in the table for $f(x)$ to find a value for x that will satisfy the equation $f(x)=0$. So $f(x)=0$ when $x=3$. So, $g(f(x))=2$ when $x=3$.

Composition by Graphs

Given the graphs of the functions for $f(x)$ *and* $g(x)$ below, find the values of $f(g(2))$ and $g(f(-1))$





Solution:

To find $f(g(2))$, we need to work from the inside out. We will therefore evaluate $g(2)$ first. To do so, we will look at the graph for $g(x)$ and find the value of the function when x = 2.
So, $g(2)=4$. This value then gets plugged into the function $f(x)$. So $f(g(2))= f(4)$. Next, we will evaluate $f(4)$ by examining the value you get from the graph for $f(x)$ when x = 4, where we get the value of 5. Therefore, $f(g(2))=5$.

Now try $g(f(-1))$. First find the value of $f(-1)$ from the table. You should get $f(-1)=4$. Now use that value to evaluate the function $g(x)$, from the table where $x=4$. You should get that $g(4)=3$. So $g(f(-1))=3$.

How about we see if you really know what you are doing.

Example: Is there a value of x such that $f(g(x))=3$. Justify your answer.

Solution:

Since we do not know the value for x, we will start by working from the outside in. So we will be looking for values of x that satisfy $f(x)=3$. Essentially these will be y-values for $f(x)$ that are 3. These occur at x = 0 and x = 3. Next we look for values of x that y-values of $g(x)$ that are either 0 or 3. Obvious values occur for $g(x)$ when x = 0 and x = 4. There is, however, another value that exists between 1 and 2. Examining the slope and y-intercept of the line on the left side of the piecewise function for $g(x)$, it has the equation y = 2x. If you replace y with the value of 3, we get $3=2x$, and $x=\frac{3}{2}$. So, in summary, there are values of x that satisfy $f(g(x))=3$. They are $x=0, \frac{3}{2}, \text{ and } 4$. These will give values where $g(x)=0 \text{ or } g(x)=3$, which makes $f(x)=3$.